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2001 J. Phys. A: Math. Gen. 34 5297

(http://iopscience.iop.org/0305-4470/34/25/311)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 5297–5309 PII: S0305-4470(01)20227-3

N-soliton solutions for second-harmonic generation with account of the Kerr effect

H Steudel[1] **and A A Zabolotskii**[2]

[1] Institut für Physik der Humboldt-Universität, Invalidenstraße 110, 10115 Berlin, Germany
[2] Institute of Automation and Electrometry, Siberian Branch of the Russian Academy of Sciences, 630090 Novosibirsk, Russia

E-mail: steudel@physik.hu-berlin.de and zabolotskii@iae.nsk.su

Received 8 December 2000, in final form 21 May 2001
Published 15 June 2001
Online at stacks.iop.org/JPhysA/34/5297

Abstract

For second-harmonic generation the optical Kerr effect causes a cross-phase modulation between ground and harmonic waves as well as self-phase modulations of both waves. When only cross-phase modulation is taken into account this system proves to be integrable by the inverse scattering method. Here Darboux/Bäcklund transformations are established, and the resulting phenomenology of soliton solutions is discussed and depicted.

PACS numbers: 4265K, 0230J, 4265T

1. Introduction

Frequency doubling or second-harmonic generation (SHG) was the very first process realized in nonlinear optics [1] and, nowadays, has become a standard tool in optical laboratories. Its theoretical treatment under stationary conditions in one space dimension was developed very early [2]. Nevertheless, SHG still provides important aspects both from the practical and from the theoretical points of view. Here we will not consider focusing or related transversal effects but take into account one space dimension only. For short pulses the walk-off of the pulses at both frequencies is of importance. The SHG equations (with no Kerr-type terms) are ‘S-integrable’, i.e. integrable by the spectral transform or inverse scattering method [3–7]. Further restriction to pure amplitude modulation leads to ‘C-integrability’. This means that the system is *integrable by change of variables*. Its solution can be given explicitly [8–11]. Without reference to the Liouville equation particular analytical solutions were given for this system in [12]. It is of importance to notice that for SHG equations without Kerr terms no soliton-type solutions exist.

The optical Kerr effect causes self-phase modulation of both participating waves at the fundamental and at the harmonic frequencies, and it causes cross-phase modulation of both components. The complete SHG–Kerr equations, (see (1) and (2) below) are probably not integrable. When, however, cross-phase modulation is taken into account while self-phase modulation is neglected this leads to a new S-integrable system, see (9) and (10). Remarkably and in contrast to the 'pure' SHG equations, this system admits a rich phenomenology of soliton solutions.

In section 2 the basic equations are introduced and transformed to a convenient form. The equations of motion are integrability conditions of some *linear system* as established in section 3. The Darboux/Bäcklund transformation for our system is developed in section 4, and its N-fold iteration is given in section 5. Here we make use of *Vandermonde-like determinants*, whose definition is repeated in the appendix. In section 6 we study monochromatic wave solutions, which are then used as 'seed solutions'. The phenomenology of soliton solutions is developed and depicted in section 7.

It is worth mentioning that there is a strong correspondence between SHG and two-photon absorption (or, more exactly, *degenerate two-photon propagation*) in the limit of low excitation (cf [6]). The harmonic wave then corresponds to the excitation amplitude of the upper level, and cross-phase modulation corresponds to a population-dependent refraction number on one hand and optical Stark shift on the other. The results of the present paper also apply to such a physical system.

2. Basic equations

SHG in a Kerr medium is described by the equations

$$\partial_\chi q_1 = -2q_2 q_1^* + \mathrm{i}(\kappa_{11}|q_1|^2 + \kappa_{21}|q_2|^2)q_1 \tag{1}$$

$$\partial_\tau q_2 = q_1^2 + \mathrm{i}(\kappa_{12}|q_1|^2 + \kappa_{22}|q_2|^2)q_2. \tag{2}$$

The star denotes complex conjugation. Here it is assumed that the ground wave q_1 and the harmonic wave q_2 are propagating with different group velocities v_1 and v_2 respectively. Then the characteristic coordinates χ, τ are connected to the laboratory space and time coordinates x, t by

$$\chi = v(-t + x/v_2) = -v\tau_2 \qquad \tau = v(t - x/v_1) = v\tau_1 \tag{3}$$

where the parameter v describing the group velocity mismatch is given by

$$v = (1/v_2 - 1/v_1)^{-1}. \tag{4}$$

For definiteness we assume $v_1 > v_2$. However, the translation of any statement or solution to the opposite case becomes obvious. The coefficients κ_{ik} are real numbers, and the corresponding terms in (1) and (2) describe the optical Kerr effect. The SHG equations with the Kerr terms omitted are integrable [3]. It would be of interest to ask whether the Kerr terms necessarily destroy integrability or not. Here we will demonstrate that there is integrability under the assumption that self-interaction terms are neglected; i.e., $\kappa_{11} = \kappa_{22} = 0$, κ_{12} and κ_{21} arbitrary real.

Generally, all four κ-coefficients in (1) and (2) are expected to be of the same order of magnitude (cf [13]). As is well known (see, e.g., [14], section 2.2.3) in a two-level model the Kerr coefficient changes sign at exact resonance. We cannot work at exact resonance because then the excitation of the medium would become important. However, by an appropriate doping of the crystal combined with a proper choice of frequency it should be possible to achieve a partial compensation of self-phase modulation terms. In summary we may conclude that our assumption to neglect self-phase modulation is not actually unrealistic. Clearly, these terms could be taken into account afterwards by some perturbational method.

We wish to bring (1) and (2), with $\kappa_{11} = \kappa_{22} = 0$, to a convenient normal form. As a first step we show that κ_{21} can be transformed to zero. Due to the conservation law

$$\partial_\chi |q_1|^2 + 2\partial_\tau |q_2|^2 = 0 \tag{5}$$

which is easily verified, there is a function $F(\chi, \tau)$ such that

$$\partial_\tau F = |q_1|^2 \qquad \partial_\chi F = -2|q_2|^2. \tag{6}$$

Then, under the transformation

$$\tilde{q}_1 = \exp[\mathrm{i}\kappa_{21} F/2] q_1 \qquad \tilde{q}_2 = \exp[\mathrm{i}\kappa_{21} F] q_2 \tag{7}$$

we obtain equations of the same structure as (1), (2) but with

$$\tilde{\kappa}_{12} = \kappa_{12} + \kappa_{21} \qquad \tilde{\kappa}_{21} = 0. \tag{8}$$

With other words, we may take $\kappa_{21} = 0$ in (1), (2) without loss in generality. The sign of κ_{12} changes under the transformation $q_1 \leftrightarrow q_1^*, q_2 \leftrightarrow q_2^*$. Its numerical value changes under the transformation $\tilde{\tau} = \tau/a, \tilde{\chi} = \chi/a, \tilde{q}_1 = aq_1, \tilde{q}_2 = aq_2, \tilde{\kappa}_{12} = \kappa_{12}/a$. Here, without loss in generality, we will take $\tilde{\kappa}_{12} = 2$. If, finally, we apply a transformation $\tilde{q}_1 = q_1 \exp(\mathrm{i}\chi), \tilde{q}_2 = q_2 \exp(2\mathrm{i}\chi)$ and, afterward, omit the tildes we arrive at

$$\partial_\chi q_1 = \mathrm{i}q_1 - 2q_2 q_1^* \tag{9}$$

$$\partial_\tau q_2 = q_1^2 + 2\mathrm{i}|q_1|^2 q_2 = -\mathrm{i}q_1(\mathrm{i}q_1 - 2q_2 q_1^*) \tag{10}$$

and this is the system of equations that will be investigated below. This last step physically corresponds to a redefinition of the split carrier waves. All further investigations refer to the normal form (9), (10) of our dynamical system.

We notice that the equations are invariant under the scale transformation $\tilde{q}_1 = bq_1,\ \tilde{\tau} = t/b^2$.

Next we will establish that it is indeed integrable by the spectral transform (or inverse scattering) method.

3. The linear system and the Riccati equations

The above equations (9) and (10) prove to be the integrability conditions of the partial differential equations

$$\partial_\chi \phi = \zeta \begin{pmatrix} -\mathrm{i}\zeta & 2q_2^* \\ 2q_2 & \mathrm{i}\zeta \end{pmatrix} \phi \equiv U\phi \tag{11}$$

$$\partial_\tau \phi = \frac{\mathrm{i}\zeta}{1-\zeta^2} V_1 \phi \equiv V\phi \tag{12}$$

with

$$V_1 = \begin{pmatrix} \zeta|q_1|^2 & q_1^{*2} \\ -q_1^2 & -\zeta|q_1|^2 \end{pmatrix}. \tag{13}$$

Here ϕ is a two-component column vector, $\phi = (\varphi_1, \varphi_2)^{\mathrm{T}}$. With the definition $\beta \equiv \varphi_2/\varphi_1$ from (11)–(13) we obtain the Riccati equations

$$\partial_\chi \beta = 2\zeta(\mathrm{i}\zeta\beta + q_2 - q_2^*\beta^2) \tag{14}$$

$$\partial_\tau \beta = \frac{\mathrm{i}\zeta}{\zeta^2 - 1}(2\zeta|q_1|^2\beta + q_1^2 + q_1^{*2}\beta^2) \tag{15}$$

and the integrability condition for these simultaneous Riccati equations as well is equivalent to our system (9), (10). For later use the following observations are worth noting:

(i) If β solves (14) and (15) then $1/\beta^*$ solves the same equations with ζ replaced by $-\zeta^*$.
(ii) For ζ real one may easily derive equations of the structure

$$\partial_\chi(\beta^*\beta) = \text{something} \times (\beta^*\beta - 1) \qquad \partial_\tau(\beta^*\beta) = \text{something} \times (\beta^*\beta - 1) \tag{16}$$

so that if $|\beta| = 1$ is true anywhere the same holds everywhere.

Equation (11) defines a 2×2-scattering problem of second order in the *spectral parameter* ζ. Remarkably, the same scattering problem appeared in connection with the derivative nonlinear Schrödinger (DNLS) equation [15].

4. The one-step Darboux/Bäcklund transformation

The term *Darboux transformation* denotes a method to derive from one solution of some scattering problem—(11) in our case—with specified potential a new solution with some transformed potential. When extended to a simultaneous linear system—(11), (12) in our case—Darboux transformations become *Bäcklund transformations*, which then include a transformation of solutions to the related nonlinear evolution equation(s)—in our case (1) and (2). Darboux transformations for the scattering problem defined by (11) were constructed by several authors [16–19]. Here we will follow the procedure developed in [20] and in this way establish Bäcklund transformations to our system (11), (12). Iteration of this procedure leads to a hierarchy of solutions. We may start from *vacuum* ($q_1 = q_2 = 0$) or from monochromatic waves as the *seed solutions*. First let us consider a scattering problem slightly more general than (11)

$$\begin{aligned} &\partial_\chi\phi = \zeta(\zeta J + Q)\phi \\ &J = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} \qquad Q = \begin{pmatrix} 0 & q(\chi) \\ r(\chi) & 0 \end{pmatrix} \end{aligned} \tag{17}$$

and, for a moment, omit the τ-dependence. The corresponding Riccati equation then reads

$$\partial_\chi\beta = \zeta(2\mathrm{i}\zeta\beta + r - q\beta^2). \tag{18}$$

Theorem (Darboux transformation). *Given one particular solution* $\{\phi_1(\chi), \zeta_1, q(\chi), r(\chi)\}$ *to (11) or* $\{\beta_1(\chi), \zeta_1, q(\chi), r(\chi)\}$ *to (18) we define the matrix*

$$M_1(\zeta) = \begin{pmatrix} \beta_1\zeta/\zeta_1 & -1 \\ -1 & \alpha_1\zeta/\zeta_1 \end{pmatrix} \qquad \alpha_1 \equiv 1/\beta_1. \tag{19}$$

Then from any solution $\{\phi, \zeta, q, r\}$ *to (11) a new solution* $\{\tilde\phi, \zeta, \tilde q, \tilde r\}$ *is found by the Darboux transformation*

$$\tilde\phi = M_1\phi \qquad \tilde q = \beta_1(\beta_1 q - 2\mathrm{i}\zeta_1) \qquad \tilde r = \alpha_1(\alpha_1 r + 2\mathrm{i}\zeta_1). \tag{20}$$

The proof is easy and straightforward and may be omitted here.

Commutativity

When we apply two Darboux transformations with the parameters ζ_1 and ζ_2 successively and denote by β_{12} the corresponding β-function at the second step we may derive an auxilary relation,

$$\beta_1\beta_{12} = \beta_2\beta_{21} = \frac{\beta_2\zeta_2 - \beta_1\zeta_1}{\beta_1\zeta_2 - \beta_2\zeta_1}. \tag{21}$$

By use of this relation it can easily be confirmed that Darboux transformations commute.

For later use it is of importance to notice that the matrix function M alternatively could be characterized by the properties

$$(1)\quad M_1(\zeta_1)\phi_1 \equiv 0 \qquad (2)\quad M_1(0) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{22}$$

Principally, there is some arbitrariness in the choice of the normalization (2).

Reduction. If $r = q^*$, ζ_1 real and β_1 is chosen of modulus 1 then it follows that $\alpha_1 = \beta_1^*$, $\tilde{r} = \tilde{q}^*$. That is, the symmetry $r = q^*$ is conserved.

Corollary (Bäcklund transformation). *Given one particular simultaneous solution* $\{\beta_1(\chi,\tau), \zeta_1, q_1(\chi,\tau), q_2(\chi,\tau)\}$ *to (11) and (12) with* ζ_1 *real and* $|\beta_1| = 1$ *the subsequent formulae together with (19) define a transformation of any solution* $\{\phi, \zeta, q_1, q_2\}$ *to (11) and (12) to a new solution* $\{\tilde{\phi}, \zeta, \tilde{q}_1, \tilde{q}_2\}$,

$$\tilde{\phi} = M_1\phi \tag{23}$$

$$\tilde{q}_1 = \frac{\alpha_1 q_1 + \zeta_1 q_1^*}{\sqrt{1-\zeta_1^2}} \tag{24}$$

$$\tilde{q}_2 = \alpha_1(\alpha_1 q_2 + \mathrm{i}\zeta_1). \tag{25}$$

The proof can be executed by direct verification.

As a consequence, equations (24) and (25) define a transformation

$$\{q_1, q_2\} \to \{\tilde{q}_1, \tilde{q}_2\} \tag{26}$$

of solutions to (9) and (10).

5. The N-fold Darboux/Bäcklund transform

Now we return to the spectral problem (11) with no reduction so far and suppressing the τ-dependence. Given N solutions $\{\beta_k, \zeta_k\}$, $k = 1, \ldots, N$ to (14) with fixed *potentials* $q(\chi)$, $r(\chi)$ then for the N-fold Darboux transform the wavefunction is an Nth-order polynomial in ζ,

$$\phi^{[N]} = M_N(\zeta)\phi \qquad M_N(\zeta) = \sum_{k=0}^{N} P_k\zeta^k. \tag{27}$$

From the iteration of (19) the coefficients P_k acquire the structure

$$P_k = \begin{cases} \begin{pmatrix} 0 & p_k \\ s_k & 0 \end{pmatrix} & N-k \quad \text{odd} \\ \begin{pmatrix} p_k & 0 \\ 0 & s_k \end{pmatrix} & N-k \quad \text{even} \end{cases} \tag{28}$$

and thus we may write

$$M_N(\zeta) = \begin{pmatrix} \mathcal{P}_N(\zeta) & \mathcal{P}_{N-1}(\zeta) \\ \mathcal{S}_{N-1}(\zeta) & \mathcal{S}_N(\zeta) \end{pmatrix} \tag{29}$$

with

$$\mathcal{P}_{2l+1}(\zeta) \equiv \sum_{m=0}^{l} p_{2m+1}\zeta^{2m+1} \qquad \mathcal{P}_{2l}(\zeta) \equiv \sum_{m=0}^{l} p_{2m}\zeta^{2m} \tag{30}$$

and the same equations with $p, \mathcal{P}$ replaced by $s, \mathcal{S}$. P_0 becomes off-diagonal for N odd and diagonal for N even, and we obtain $p_0 = s_0 = \pm 1$. Because there is an arbitrary constant overall factor we may choose

$$p_0 = s_0 = -1. \tag{31}$$

From $M_1(\zeta_1)\phi_1 = 0$ (cf (22)) together with commutativity it follows that

$$M_N(\zeta_j)\phi_j = 0 \qquad j = 1, \ldots, N. \tag{32}$$

Thus we arrive at two complete systems of linear algebraic equations separately for p_k and for s_k, $k = 1, \ldots, N$. We have to distinguish whether N is odd or even, and will write down the coefficients p_N and p_{N-1} explicitly because we will see below that it is just these entities which are required, and we will use the notation of Vandermonde-like determinants (see the appendix). Determinants of such a type have been used by several authors [21, 22]. To the best of our knowledge their structure was first investigated in a systematic way in [5, 23].

$N = 2n + 1$:

$$\sum_{k=1}^{n} p_{2k}\zeta_j^{2k} + \sum_{k=0}^{n} p_{2k+1}\zeta_j^{2k+1}\alpha_j = 1 \tag{33}$$

$$\sum_{k=1}^{n} s_{2k}\zeta_j^{2k} + \sum_{k=0}^{n} s_{2k+1}\zeta_j^{2k+1}\beta_j = 1 \tag{34}$$

$$p_{2n+1} = \frac{\mathcal{V}_{n+1,n}(1_j;\, \zeta_j\alpha_j|\zeta_j^2)}{\mathcal{V}_{n,n+1}(\zeta_j^2;\, \zeta_j\alpha_j|\zeta_j^2)} \tag{35}$$

$$p_{2n} = (-1)^{n-1}\frac{\mathcal{V}_{n,n+1}(1_j;\, \zeta_j\alpha_j|\zeta_j^2)}{\mathcal{V}_{n,n+1}(\zeta_j^2;\, \zeta_j\alpha_j|\zeta_j^2)} \tag{36}$$

$N = 2n$:

$$\sum_{k=1}^{n} p_{2k}\zeta_j^{2k} + \sum_{k=1}^{n} p_{2k-1}\zeta_j^{2k-1}\beta_j = 1 \tag{37}$$

$$\sum_{k=1}^{n} s_{2k}\zeta_j^{2k} + \sum_{k=1}^{n} s_{2k-1}\zeta_j^{2k-1}\alpha_j = 1 \tag{38}$$

$$p_{2n} = (-1)^{n-1}\frac{\mathcal{V}_{nn}(1_j;\, \zeta_j\beta_j|\zeta_j^2)}{\mathcal{V}_{nn}(\zeta_j^2;\, \zeta_j\beta_j|\zeta_j^2)} \tag{39}$$

$$p_{2n-1} = -\frac{\mathcal{V}_{n+1,n-1}(1_j;\, \zeta_j\beta_j|\zeta_j^2)}{\mathcal{V}_{nn}(\zeta_j^2;\, \zeta_j\beta_j|\zeta_j^2)}. \tag{40}$$

Here j runs from unity to N, and the notation '$(1_j; \ldots)$' means that N arguments are put equal to unity. In both cases—N odd or even—s_N and s_{N-1} are easily found from the respective formulae for p_N and p_{N-1} because it holds generally that

$$s_k = p_k(\alpha \longleftrightarrow \beta). \tag{41}$$

Let us write the transformed spectral problem in the form

$$\phi_x^{[N]} = \zeta(\zeta J + Q^{[N]})\phi^{[N]}. \tag{42}$$

Substitution of (27) into (11) and (42) and comparison of powers in ζ leads to

$$\begin{array}{lllll} & [P_0, J] & & = 0 & \\ & [P_1, J] & + P_0 Q - Q^{[N]} P_0 & = 0 & \\ P_{k-1,x} & + [P_{k+1}, J] & + P_k Q - Q^{[N]} P_k & = 0 & \qquad k = 1..N-1 \\ P_{N-1,x} & & + P_N Q - Q^{[N]} P_N & = 0 & \end{array} \tag{43}$$

and from the second equation of this system we obtain

$$q^{[N]} = \frac{p_N q + 2\mathrm{i} p_{N-1}}{s_N} \qquad r^{[N]} = \frac{s_N r - 2\mathrm{i} s_{N-1}}{p_N}. \tag{44}$$

Reduction $r = q^*$. Then one has to take the eigenvalues as real or as complex conjugate pairs, $\zeta_l = \zeta_k^*$, and to choose

(i) $|\beta_j| = 1$ for real ζ_j, and
(ii) $\beta_l = 1/\beta_k^* = \alpha_k^*$ when $\zeta_l = \zeta_k^*$.

Then we obtain either $s_j = p_j^*$ or $s_k = p_l^*$ respectively and therefore $\mathcal{S}_k(\zeta) = \mathcal{P}_k(\zeta^*)^*$. Consequently the required symmetry is conserved.

If now we assume that $q_1(\chi, \tau), q_2(\chi, \tau)$ are solutions to the equations (1) and (2) and β_k are simultaneous solutions to the Riccati equations (14) and (15) then we know that there exists an N-step Bäcklund transform, and by the above formulae it is determined uniquely with the identification $q = 2q_2^*,\ r = 2q_2$. To complete the transformation we would like to have a formula for $q_1^{[N]}$. As a preparation it is useful to obtain formulae for the polynomials $\mathcal{P}_N(\zeta), \mathcal{P}_{N-1}(\zeta)$ at an arbitrary argument ζ.

For $N = 2n + 1$ we may take the formal equation

$$-\mathcal{P}_{2n+1}(\zeta) + \sum_{k=0}^{n} p_{2k+1}\zeta^{2k+1} = 0 \tag{45}$$

together with (33) and read this as a system of $2n + 2$ equations for determining the $2n + 2$ unknowns $(\mathcal{P}_{2n+1}(\zeta), p_j)$. Then, by use of Cramer's rule, we arrive at
$N = 2n + 1$

$$\mathcal{P}_{2n+1}(\zeta) = -\frac{\mathcal{V}_{n+1,n+1}(0, 1_j; \zeta, \zeta_j\alpha_j|\zeta^2, \zeta_j^2)}{\mathcal{V}_{n,n+1}(\zeta_j^2; \zeta_j\alpha_j|\zeta_j^2)} \tag{46}$$

and in quite an analogous way we write

$$-\mathcal{P}_{2n}(\zeta) + \sum_{k=1}^{n} p_{2k}\zeta^{2k} = 1 \tag{47}$$

and combine this equation with the system (33). Then we find

$$\mathcal{P}_{2n}(\zeta) = -\frac{\mathcal{V}_{n+1,n+1}(1, 1_j; 0, \zeta_j\alpha_j|\zeta^2, \zeta_j^2)}{\mathcal{V}_{n,n+1}(\zeta_j^2; \zeta_j\alpha_j|\zeta_j^2)}. \tag{48}$$

Similarly we proceed for even N.
$N = 2n$

$$\mathcal{P}_{2n-1}(\zeta) = -\frac{\mathcal{V}_{n+1,n}(0, 1_j; \zeta, \zeta_j\beta_j|\zeta^2, \zeta_j^2)}{\mathcal{V}_{n,n}(\zeta_j^2; \zeta_j\beta_j|\zeta_j^2)} \tag{49}$$

$$\mathcal{P}_{2n}(\zeta) = -\frac{\mathcal{V}_{n+1,n}(1, 1_j; 0, \zeta_j\beta_j|\zeta^2, \zeta_j^2)}{\mathcal{V}_{n,n}(\zeta_j^2; \zeta_j\beta_j|\zeta_j^2)}. \tag{50}$$

Now we wish to obtain the transformation of the matrix V_1 in (12). By differentiation of (27) and substitution of ϕ_τ by (12) and $\phi_\tau^{[N]}$ by

$$\partial_\tau\phi^{[N]} = \frac{\mathrm{i}\zeta}{1 - \zeta^2} V_1^{[N]}\phi^{[N]} \tag{51}$$

and multiplication by $(1 - \zeta^2)$ we obtain

$$(1 - \zeta^2)\partial_\tau M_N = \mathrm{i}\zeta(V_1^{[N]} M_N - M_N V_1) \tag{52}$$

with

$$M_N(\zeta) = \begin{pmatrix} \mathcal{P}_N(\zeta) & \mathcal{P}_{N-1}(\zeta) \\ \mathcal{P}^*_{N-1}(\zeta) & \mathcal{P}^*_N(\zeta) \end{pmatrix}. \tag{53}$$

Taking this equation at $\zeta = 1$ we arrive at

$$V_1^{[N]} = M_N(1) V_1 M_N^{-1}(1). \tag{54}$$

The normalized matrix

$$\begin{aligned} \bar{M}_N &\equiv M_N(1)/\sqrt{D_N} \\ D_N &= \det M_N(1) \end{aligned} \tag{55}$$

may be written

$$\bar{M}_N = \begin{pmatrix} m_N & m_{N-1} \\ m^*_{N-1} & m^*_N \end{pmatrix}. \tag{56}$$

N odd:

$$m_N = \mathcal{P}_N(1)/\sqrt{D_N} \qquad m_{N-1} = \mathcal{P}_{N-1}(1)/\sqrt{D_N}. \tag{57}$$

N even:

$$m_N = [\mathcal{P}_N(1) - 1]/\sqrt{D_N} \qquad m_{N-1} = \mathcal{P}_{N-1}(1)/\sqrt{D_N}. \tag{58}$$

M_N is a unitary unimodular matrix. It can easily be checked that $V_1^{[N]}$ takes the same form as V_1 in (13)with

$$q_1^{[N]} = m^*_N q_1 - m^*_{N-1} q^*_1. \tag{59}$$

The transformation of q_2 is given by formula (44), from which due to symmetry and with recalling $q = 2q_2^*$ we obtain

$$q_2^{[N]} = \frac{p^*_N q_2 - \mathrm{i} p^*_{N-1}}{p_N}. \tag{60}$$

Now the formulae for the N-fold Bäcklund transformation are complete.

6. Simple seed solutions and the related solutions to the Riccati equations

When starting from vacuum $q_1 = q_2 = 0$ (14), (15) leads to the solution

$$\beta = C \exp[2\mathrm{i}\zeta^2\chi] \tag{61}$$

independent of τ. Then, according to (24) and (25), the one-step Bäcklund transform is a monochromatic harmonic wave depending on χ only while the ground wave is vanishing. Iteration generally leads to solutions with q_2 a function of χ alone and $q_1 \equiv 0$.

Let us look for a seed solution different from vacuum. It is easy to see that the monochromatic waves

$$\begin{aligned} q_1 &= a_1 \mathrm{e}^{\mathrm{i}(k\chi - w\tau)} \\ q_2 &= \mathrm{i} a_2 \mathrm{e}^{2\mathrm{i}(k\chi - w\tau)} \end{aligned} \tag{62}$$

fulfil (9), (10) provided that it holds that

$$k = 1 - 2a_2 a_1^*/a_1 \qquad w = k a_1^2/2a_2 \tag{63}$$

with k, w real. Omitting an arbitrary phase factor from a_1 we may take the amplitudes a_1, a_2 as real (but admit negative values a_2).

Before applying a Bäcklund transformation to such a seed solution it is of interest to study the nature of these solutions itself and, in particular, to require their stability. When in (62) we replace a_k by $a_k + b_k(\chi, \tau) + \mathrm{i}c_k(\chi, \tau)$, $k = 1, 2$ with $b_k^2, c_k^2 \ll a_k^2$ then substitution into (9) and (10) taking only first-order terms leads to linear partial differential equations:

$$\begin{aligned} b_{1\chi} &= -4a_2c_1 + 2a_1c_2 \\ c_{1\chi} &= -2a_1b_2 \\ b_{2\tau} &= 2a_1c_1 - (a_1^2/a_2)c_2 \\ c_{2\tau} &= 2a_1(2a_2 - 1)b_1 + (a_1^2/a_2)b_2. \end{aligned} \tag{64}$$

Then an ansatz with the common factor $\exp[\mathrm{i}(K\chi - W\tau)]$ leads to the dispersion relation:

$$\text{either} \quad K = 0 \qquad W \text{ arbitrary} \tag{65}$$

$$\text{or} \qquad K = \frac{8a_1^2a_2^2(1 - a_2)W}{a_1^4 - a_2^2W^2}. \tag{66}$$

From this dispersion relation we may conclude that the monochromatic wave solutions are, at least, 'stable' in the sense that an exponential growth of infinitesimal perturbations does not take place.

From (14) and (15) with the specifications (62) and (63) and taking

$$\beta_1 = \gamma_1 \exp[2\mathrm{i}(k\chi - w\tau)] \tag{67}$$

we obtain Riccati equations with constant coefficients for γ_1, which are easily solved,

$$\gamma_1 = \frac{-(c + bC)\tanh[d(\chi - v\tau)] + \mathrm{i}Cd}{(cC + b)\tanh[d(\chi - v\tau)] + \mathrm{i}d} \tag{68}$$

$$b = (\zeta_1^2 - k) \qquad c = 2a_2\zeta_1 \tag{69}$$

$$d = \sqrt{c^2 - b^2} \qquad v = \frac{a_1^2}{2a_2(1 - \zeta_1^2)}. \tag{70}$$

7. The phenomenology of solitons

Now we are able to study the properties of N-soliton solutions and, in particular, to depict them by computer graphics. As we shall see, there is already a rich manifold of one-soliton solutions.

One-soliton solutions

In (68) C is an integration constant. When $\zeta_1 = \xi_1$ real, k chosen such that $d^2 = (\xi_1^2 - k^2)(1 - \xi_1^2) > 0$ and C taken of modulus 1 then γ_1 becomes of modulus 1 as well. By use of a shift in space C may be transformed to +1 or -1 so that—with respect to a one-soliton solution—we may put $C = \pm 1$ for convenience. From (19), (24), (25) and (67) then we may derive simple intensity formulae,

$$|q_1^{[1]}|^2 = |a_1|^2 \frac{1 + \xi_1^2 + 2\xi_1 \mathrm{Re}\,\gamma_1}{|1 - \xi_1|^2} \tag{71}$$

$$|q_2^{[1]}|^2 = a_2^2 + \xi_1^2 + 2a_2\xi_1 \mathrm{Re}\,\gamma_1 \tag{72}$$

with the respective asymptotic values

$$|q_1^{[1]}|^2_{\mathrm{asy}} = |a_1^2(a_2^{-1} - 1)| \tag{73}$$

$$|q_2^{[1]}|^2_{\mathrm{asy}} = (a_2 - 1)^2. \tag{74}$$

The auxiliary function γ_1 asymptotically becomes constant and, clearly, in the asymptotic region a Bäcklund transformation causes a mapping among the set of monochromatic wave

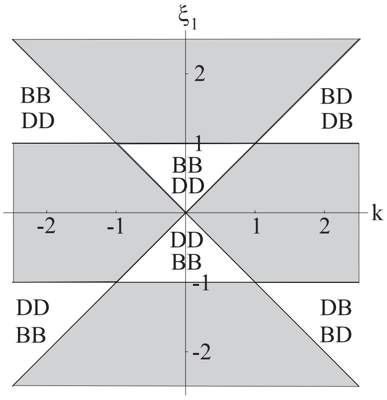

Figure 1. Six regions of solitons in the $k - \xi_1$ plane. B = 'bright', D = 'dark'. The first letter refers to the ground wave, the second to the harmonic wave.

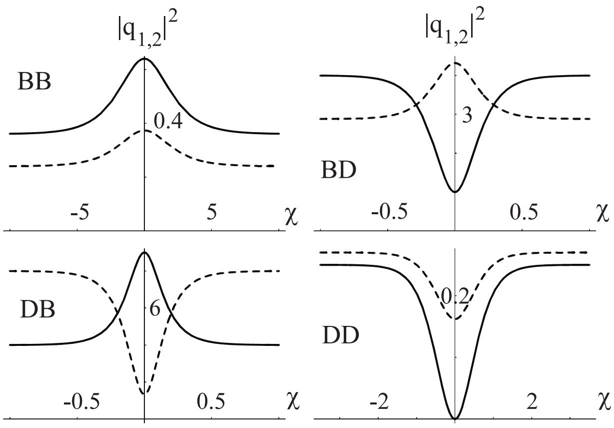

Figure 2. Four types of soliton. BB, $a_1 = 0.4, k = 0.2, \xi_1 = 0.4$; BD, $a_1 = 1.2, k = 3, \xi_1 = 2$; DB, $a_1 = 2, k = 3, \xi_1 = -2$; DD, $a_1 = 0.9, k = -2, \xi_1 = -1.5$. Dashed and full curves indicate the ground and harmonic waves respectively.

solutions (62), (63) up to constant phase factors. This mapping can be characterized by the rules (k, w) change sign, a_1^2/a_2 are invariant and $|a_1|$ transforms to $|1 - a_2|$. Consequently, after two steps the asymptotic state is restored (up to constant phase factors).

The condition $d^2 = (\xi_1^2 - k^2)(1 - \xi_1^2) > 0$ defines six admissible regions in the $k - \xi_1$ plane (see figure 1). For both waves the solutions might appear as *bright* (B) or as *dark* (D) solitons and all four combinations may occur. For brevity we use the notation 'BD' with the meaning 'bright soliton of the fundamental wave combined with a dark soliton of the harmonic wave' and, correspondingly, DB, BB, DD. In figure 2 examples of all these four types of combined soliton are depicted.

Two-soliton solutions and breathers

By choosing $\zeta_1 = \xi_1, \zeta_2 = \xi_2$ both real and combined with admissible values k_1, k_2 we may establish β_1, β_2 corresponding to (67) and (68). Then the two-soliton solutions are determined by (39), (40), (49), (50), (58) , (59) and (60) specified to $n = 1$, $N = 2$, and the result can be

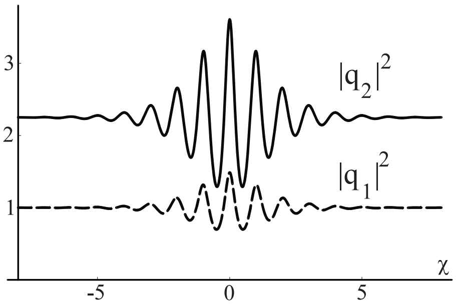

Figure 3. Breather-type solution at fixed $\tau . a_1 = 1$, $k = -2$, $\zeta_1 = 0.2 + \mathrm{i}$, $\zeta_2 = 0.2 - \mathrm{i}$.

written in the form

$$q_1^{[2]} = \frac{K_1^* + \zeta_1^* \zeta_2^* K_2^* + K_3^*}{K_2^* \sqrt{1-\zeta_1^*}\sqrt{1-\zeta_1^*}} \mathrm{e}^{\mathrm{i}(-k\chi + w\tau)} \tag{75}$$

$$q_2^{[2]} = \mathrm{i} \frac{K_1^* a_2 + K_3^*}{K_2^*} \mathrm{e}^{2\mathrm{i}(-k\chi + w\tau)} \tag{76}$$

by use of the abbreviations

$$\begin{aligned} K_1 &= \zeta_1\gamma_1 - \zeta_2\gamma_2 \\ K_2 &= \zeta_1\gamma_2 - \zeta_2\gamma_1 \\ K_3 &= \zeta_1^2 - \zeta_2^2. \end{aligned} \tag{77}$$

For ζ_1, ζ_2 taken real and consistent with figure 1 these formulae give an interacting two-soliton solution while for $\zeta_2 = \zeta_1^*$ a breather-type solution appears. In figure 3 a snapshot of such a breather-type solution is depicted.

Higher N-soliton collisions

As mentioned above, the same scattering problem (11) also appeared for the DNLS equation. Thus the Darboux transformation technique is the same for both problems. For the DNLS equation computer pictures up to $N = 8$ were generated in [20], and we could do the same for our present problem. The present situation, however, is more complicated because there are two components instead of one. As an example here we depict only a four-soliton solution: figure 4 presents three-dimensional plots of the intensities with two different choices of the scale in order to see both the asymptotic structure (left) and the collision centre (right).

8. Conclusion

We have found a rich manifold of soliton solutions for SHG with cross-phase modulation due to the optical Kerr terms being included. On the other hand, it is well known that for the ‘pure’ SHG equations without Kerr terms no soliton solutions exist. Our approach giving N-soliton formulae in terms of Vandermonde-like determinants proves to be well suited for numerical evaluation.

As a final remark we note that this approach somewhat parallels the systematic use of Wronskians for establishing N-soliton formulae (see, e.g., [24, 25]). One advantage of the formulae in terms of Vandermonde-like determinants is that no derivatives of high-order determinants appear as is typical for the approach using Wronskians.

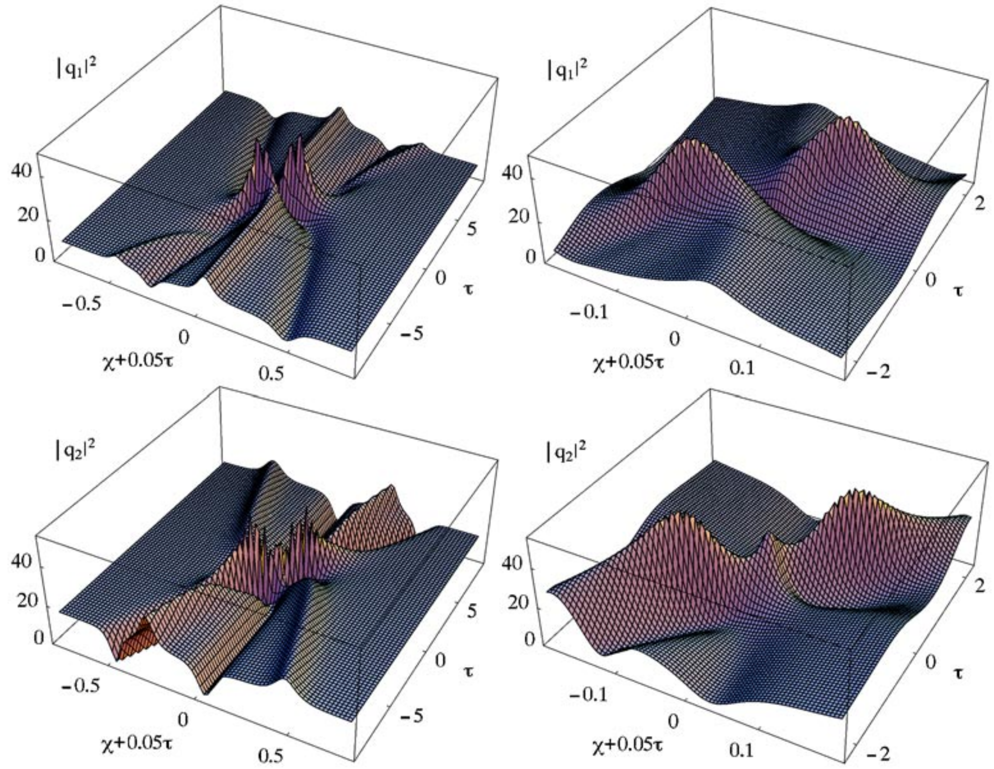

Figure 4. Example of a four-soliton solution. $a_1 = 3$, $k = -7$, $\xi_{1-4} = -2, -2.2, -2.5, -3.5$. (This figure is in colour only in the electronic version)

Acknowledgments

HS wishes to thank U Peschel and D Michaelis for useful discussions and for suggestions on literature. AAZ acknowledges support by DFG grant RUS 113/89(R,S). We are grateful to one of the referees who pointed out an error in the earlier version in the transformation (7) and to the other referee for the reference to [24, 25].

Appendix. Vandermonde-like determinants

Vandermonde-like determinants are defined as follows [23],

$$\mathcal{V}_{MN}(a_r; b_r|x_r) := \begin{vmatrix} a_1 & a_1x_1 & \cdots & a_1x_1^{M-1} & b_1 & b_1x_1 & \cdots & b_1x_1^{N-1} \\ a_2 & a_2x_2 & \cdots & a_2x_2^{M-1} & b_2 & b_2x_2 & \cdots & b_2x_2^{N-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{M+N} & a_{M+N}x_{M+N} & \cdots & a_{M+N}x_{M+N}^{M-1} & b_{M+N} & b_{M+N}x_{M+N} & \cdots & b_{M+N}x_{M+N}^{N-1} \end{vmatrix} \tag{A.1}$$

where $r = 1, 2 \ldots, M+N$. These determinants have several remarkable structural properties listed in [23]. In particular there is a *reduction formula*,

$$\mathcal{V}_{MN}(a_r; b_r|x_r) = \sum_P \varepsilon_P \prod_{j=1}^{M} a_{r(j)} \prod_{k=M+1}^{M+N} b_{s(k)} \mathcal{V}_M(x_{r(1)} \ldots x_{r(M)}) \mathcal{V}_N(x_{r(N+1)} \ldots x_{s(M+N)}). \tag{A.2}$$

The sum goes over all permutations $P = (r(1), \ldots, r(M+N))$ of $(1, 2 \ldots M+N)$ such that $r(i) < r(j)$ for $i < j \leqslant N$ as well as for $N < i < j$. $\varepsilon_P = +1$ for P even or -1 for P odd. By this formula any Vandermonde-like determinant is generally expressed as a sum over binary products of genuine Vandermonde determinants $\mathcal{V}_N$ which are defined by

$$\mathcal{V}_N(x_1, \ldots, x_N) := \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{N-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{N-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_N & x_N^2 & \cdots & x_N^{N-1} \end{vmatrix}. \tag{A.3}$$

It is well known and can easily be checked directly that $\mathcal{V}_N$ can be written as a product of differences,

$$\mathcal{V}_N(x_1, \ldots, x_N) = \prod_{i>j} (x_i - x_j). \tag{A.4}$$

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